

STANLEY DEPTH OF WEAKLY POLYMATROIDAL IDEALS AND SQUAREFREE MONOMIAL IDEALS

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ABSTRACT. Let I be a weakly polymatroidal ideal or a squarefree monomial ideal of a polynomial ring S . In this paper we provide a lower bound for the Stanley depth of I and S/I . In particular we prove that if I is a squarefree monomial ideal which is generated in a single degree, then $\text{sdepth}(I) \geq n - \ell(I) + 1$ and $\text{sdepth}(S/I) \geq n - \ell(I)$, where $\ell(I)$ denotes the analytic spread of I . This proves a conjecture of the author in a special case.

1. INTRODUCTION

Let \mathbb{K} be a field and let $S = \mathbb{K}[x_1, \dots, x_n]$ be the polynomial ring in n variables over \mathbb{K} . Let M be a finitely generated \mathbb{Z}^n -graded S -module. Let $u \in M$ be a homogeneous element and $Z \subseteq \{x_1, \dots, x_n\}$. The \mathbb{K} -subspace $u\mathbb{K}[Z]$ generated by all elements uv with $v \in \mathbb{K}[Z]$ is called a *Stanley space* of dimension $|Z|$, if it is a free $\mathbb{K}[Z]$ -module. Here, as usual, $|Z|$ denotes the number of elements of Z . A decomposition \mathcal{D} of M as a finite direct sum of Stanley spaces is called a *Stanley decomposition* of M . The minimum dimension of a Stanley space in \mathcal{D} is called the *Stanley depth* of \mathcal{D} and is denoted by $\text{sdepth}(\mathcal{D})$. The quantity

$$\text{sdepth}(M) := \max \{ \text{sdepth}(\mathcal{D}) \mid \mathcal{D} \text{ is a Stanley decomposition of } M \}$$

is called the *Stanley depth* of M . Stanley [7] conjectured that

$$\text{depth}(M) \leq \text{sdepth}(M)$$

for every \mathbb{Z}^n -graded S -module M . For a reader friendly introduction to Stanley depth, we refer to [4].

Let I be a monomial ideal of S with Rees algebra $\mathcal{R}(I)$ and let $\mathfrak{m} = (x_1, \dots, x_n)$ be the graded maximal ideal of S . Then the \mathbb{K} -algebra $\mathcal{R}(I)/\mathfrak{m}\mathcal{R}(I)$ is called the *fibre ring* and its Krull dimension is called the *analytic spread* of I , denoted by $\ell(I)$. This invariant is a measure for the growth of the number of generators of the powers of I . Indeed, for $k \gg 0$, the Hilbert function $H(\mathcal{R}(I)/\mathfrak{m}\mathcal{R}(I), \mathbb{K}, k) = \dim_{\mathbb{K}}(I^k/\mathfrak{m}I^k)$, which counts the number of generators of the powers of I , is a polynomial function of degree $\ell(I) - 1$.

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In this paper we consider some linear algebraic approximations of the analytic spread of a monomial ideal. Indeed, assume that v_1, \dots, v_t are t vectors in \mathbb{Q}^n . Then they are called to be *linearly dependent* if there exist rational numbers c_1, \dots, c_t , not all zero, for which

$$c_1 v_1 + \dots + c_t v_t = 0.$$

Similarly they are *affinely dependent*, if in addition the sum of coefficients is zero:

$$\sum_{i=1}^t c_i = 0.$$

If v_1, \dots, v_t are not linearly dependent (resp. affinely dependent), then they are said to be *linearly independent* (resp. *affinely independent*). Now we associate two invariants to every monomial ideal I , which are called the rank and the affine rank of I . For every vector $\mathbf{a} = (a_1, \dots, a_n)$ of non-negative integers, we denote the monomial $x_1^{a_1} \dots x_n^{a_n}$ by $\mathbf{x}^{\mathbf{a}}$.

Definition 1.1. Let $I \subseteq S = \mathbb{K}[x_1, \dots, x_n]$ be a monomial ideal and $G(I) = \{\mathbf{x}^{\mathbf{a}_1}, \dots, \mathbf{x}^{\mathbf{a}_m}\}$ be the set of minimal monomial generators of I . The *rank* of I , denoted by $\text{rank}(I)$ is the cardinality of the largest linearly independent subset of $\{\mathbf{a}_1, \dots, \mathbf{a}_m\}$. Similarly the *affine rank* of I , denoted by $\text{arank}(I)$ is the cardinality of the largest affinely independent subset of $\{\mathbf{a}_1, \dots, \mathbf{a}_m\}$.

It is clear from Definition 1.1 that for every monomial ideal I , the inequality $\text{arank}(I) \geq \text{rank}(I)$ holds. It is known [2, Lemma 10.3.19] that if I is a monomial ideal which is generated in a single degree, then $\ell(I) = \text{rank}(I)$. The following Proposition shows that in this case we also have $\ell(I) = \text{arank}(I)$.

Proposition 1.2. *Let I be a monomial ideal, which is generated in a single degree. Then $\ell(I) = \text{rank}(I) = \text{arank}(I)$.*

Proof. It is sufficient to prove the second equality. Assume that $\text{arank}(I) = t$. Therefore, there exist integers $1 \leq i_1 < \dots < i_t \leq m$ such that the equalities

$$c_1 \mathbf{a}_{i_1} + \dots + c_t \mathbf{a}_{i_t} = 0$$

and

$$c_1 + \dots + c_t = 0,$$

with $c_i \in \mathbb{Q}$, for every $1 \leq i \leq t$, imply that $c_1 = \dots = c_t = 0$. Since I is generated in a single degree, $\mathbf{a}_{i_1}, \dots, \mathbf{a}_{i_t}$ are linearly independent over \mathbb{Q} . Indeed, assume that there exist rational numbers d_1, \dots, d_t such that

$$d_1 \mathbf{a}_{i_1} + \dots + d_t \mathbf{a}_{i_t} = 0.$$

Now for every $1 \leq j \leq t$, the sum of the components of \mathbf{a}_{i_j} is equal to k and thus, the sum of the components of

$$d_1 \mathbf{a}_{i_1} + \dots + d_t \mathbf{a}_{i_t}$$

is equal to

$$d_1 k + \dots + d_t k$$

and this shows that

$$d_1 + \dots + d_t = 0.$$

Therefore

$$d_1 = \dots = d_t = 0.$$

Hence $\mathbf{a}_{i_1}, \dots, \mathbf{a}_{i_t}$ are linearly independent over \mathbb{Q} . Therefore, $\text{rank}(I) \geq t$. Since we always have $\text{arank}(I) \geq \text{rank}(I)$, it follows that $\text{arank}(I) = \text{rank}(I)$. \square

In [5], the authors prove that if $I \subset S$ is a weakly polymatroidal ideal I (see Definition 2.1), which is generated in a single degree, then $\text{depth}(S/I) \geq n - \ell(I)$, $\text{sdepth}(S/I) \geq n - \ell(I)$ and $\text{sdepth}(I) \geq n - \ell(I) + 1$. In Section 2 we generalize this result by proving that for every weakly polymatroidal ideal I , the inequalities

$$\text{sdepth}(I) \geq n - \text{arank}(I) + 1, \quad \text{sdepth}(S/I) \geq n - \text{arank}(I)$$

and

$$\text{depth}(S/I) \geq n - \text{arank}(I)$$

hold (see Theorem 2.6).

In [8], the author conjectures that for every integrally closed monomial ideal, the inequalities $\text{sdepth}(S/I) \geq n - \ell(I)$ and $\text{sdepth}(I) \geq n - \ell(I) + 1$ hold (see Conjecture 3.1). In Section 3, we prove this conjecture for every squarefree monomial ideal which is generated in a single degree. In fact, we prove some stronger result. We show that for every squarefree monomial ideal I of the polynomial ring S , the inequalities

$$\text{sdepth}(I) \geq n - \text{rank}(I) + 1$$

and

$$\text{sdepth}(S/I) \geq n - \text{rank}(I)$$

hold (see Theorem 3.3).

2. STANLEY DEPTH OF WEAKLY POLYMATROIDAL IDEALS

Weakly polymatroidal ideals are generalization of polymatroidal ideals and they are defined as follows.

Definition 2.1 ([2], Definition 12.7.1). A monomial ideal I of $S = \mathbb{K}[x_1, \dots, x_n]$ is called *weakly polymatroidal* if for every two monomials $u = x_1^{a_1} \dots x_n^{a_n}$ and $v = x_1^{b_1} \dots x_n^{b_n}$ in $G(I)$ such that $a_1 = b_1, \dots, a_{t-1} = b_{t-1}$ and $a_t > b_t$ for some t , there exists $j > t$ such that $x_t(v/x_j) \in I$.

The aim of this section is to provide a lower bound for the depth and the Stanley depth of weakly polymatroidal ideals. As usual for every monomial u , the *support* of u , denoted by $\text{Supp}(u)$, is the set of variables, which divide u .

Lemma 2.2. *Let I be a weakly polymatroidal ideal and let $G(I) = \{u_1, \dots, u_m\}$ be the set of minimal monomial generators of I . Assume that*

$$x_1 \in \bigcup_{i=1}^m \text{Supp}(u_i).$$

Then $(I : x_1)$ is a weakly polymatroidal ideal which is minimally generated by the set

$$\mathcal{G} = \left\{ \frac{u_i}{x_1} \mid u_i \in G(I) \text{ and } x_1 \text{ divides } u_i \right\}.$$

Proof. It is clear that the ideal generated by \mathcal{G} is a weakly polymatroidal ideal. Thus, we prove that $(I : x_1)$ is generated by the set \mathcal{G} . Without loss of generality, we may assume that u_1, \dots, u_t are divisible by x_1 and u_{t+1}, \dots, u_m are not divisible by x_1 , where $1 \leq t \leq m$. Let $v_i = u_i/x_1$ ($1 \leq i \leq t$). We should prove that $(I : x_1)$ is generated by v_1, \dots, v_t . Let $v \in (I : x_1)$ be a monomial. Then $x_1 v \in I$ and so there exists $1 \leq i \leq m$ in such a way that u_i divides $x_1 v$. If $1 \leq i \leq t$, then v is divisible by v_i and therefore, $v \in (v_1, \dots, v_t)$. Hence, we may assume that $i \geq t+1$. Now u_i is not divisible by x_1 and thus $u_i | v$. Since

$$x_1 \in \bigcup_{i=1}^m \text{Supp}(u_i),$$

Definition 2.1 implies that there exists $j \geq 2$ such that $x_1 u_i / x_j \in I$. Hence, there exists $1 \leq s \leq m$, such that u_s divides $x_1 u_i / x_j$. If $t+1 \leq s \leq m$, then u_s divides u_i / x_j and thus u_s properly divides u_i , which is a contradiction, because $G(I)$ is the set of minimal monomial generators of I . It follows that $1 \leq s \leq t$. Therefore, v_s divides u_i / x_j and hence, it divides u_i . Since v is divisible by u_i , we conclude that v_s divides v . This shows that $v \in (v_1, \dots, v_t)$ and completes the proof of the lemma. \square

The following lemma shows that the affine rank of a weakly polymatroidal ideal does not increase under the colon operation with respect to the variable x_1

Lemma 2.3. *Let I be a weakly polymatroidal ideal. Then $\text{arank}((I : x_1)) \leq \text{arank}(I)$.*

Proof. If $I = (I : x_1)$, then there is nothing to prove. So assume that $I \neq (I : x_1)$. Let $G(I) = \{u_1, \dots, u_m\}$ be the set of minimal monomial generators of I . Since $I \neq (I : x_1)$, it follows that

$$x_1 \in \bigcup_{i=1}^m \text{Supp}(u_i).$$

Without loss of generality, we may assume that u_1, \dots, u_t are divisible by x_1 and u_{t+1}, \dots, u_m are not divisible by x_1 , where $1 \leq t \leq m$. Let $v_i = u_i/x_1$ ($1 \leq i \leq t$). By Lemma 2.2, the set $\{v_1, \dots, v_t\}$ is the set of minimal monomial generators of $(I : x_1)$. For simplicity we assume that \mathbf{a}_i is the exponent vector of v_i , ($1 \leq i \leq t$). Suppose that $\text{arank}((I : x_1)) = s$ and choose the monomials v_{j_1}, \dots, v_{j_s} , such that the equalities

$$c_1 \mathbf{a}_{j_1} + \dots + c_s \mathbf{a}_{j_s} = \mathbf{0}$$

and

$$c_1 + \dots + c_s = 0,$$

with $c_i \in \mathbb{Q}$, for every $1 \leq i \leq s$, imply that $c_1 = \dots = c_s = 0$. Note that for every $1 \leq i \leq t$, the exponent vector of u_i is equal to $\mathbf{a}_i + \mathbf{e}_1$, where \mathbf{e}_1 is the first vector

in the standard basis of \mathbb{Q}^n . Now assume that there exist $d_1, \dots, d_s \in \mathbb{Q}$, such that $d_1 + \dots + d_s = 0$ and

$$d_1(\mathbf{a}_{j_1} + \mathbf{e}_1) + \dots + d_s(\mathbf{a}_{j_s} + \mathbf{e}_1) = 0.$$

Therefore

$$d_1\mathbf{a}_{j_1} + \dots + d_s\mathbf{a}_{j_s} + (d_1 + \dots + d_s)\mathbf{e}_1 = 0.$$

Since $d_1 + \dots + d_s = 0$, it follows that

$$d_1\mathbf{a}_{j_1} + \dots + d_s\mathbf{a}_{j_s}.$$

By the choice of v_{j_1}, \dots, v_{j_s} , we conclude that $d_1 = \dots = d_s = 0$. Thus, $\text{arank}(I) \geq s$ and this proves our assertion. \square

In the following lemma we consider the behavior of the affine rank of an arbitrary monomial ideal under the elimination of x_1 .

Lemma 2.4. *Let I be a monomial ideal of $S = \mathbb{K}[x_1, \dots, x_n]$, such that*

$$x_1 \in \bigcup_{u \in G(I)} \text{Supp}(u).$$

Let $S' = \mathbb{K}[x_2, \dots, x_n]$ be the polynomial ring obtained from S by deleting the variable x_1 and consider the ideal $I' = I \cap S'$. Then $\text{arank}(I') + 1 \leq \text{arank}(I)$.

Proof. Let $G(I) = \{u_1, \dots, u_m\}$ be the set of minimal monomial generators of I . For simplicity we assume that \mathbf{a}_i is the exponent vector of u_i , ($1 \leq i \leq m$). Without loss of generality, we may assume that u_1, \dots, u_t are divisible by x_1 and u_{t+1}, \dots, u_m are not divisible by x_1 , where $1 \leq t \leq m$. Then the set $\{u_{t+1}, \dots, u_m\}$ is the set of minimal monomial generators of I' . Assume that $\text{arank}(I') = s$. Thus, there exist integers $t+1 \leq j_1 < j_2 < \dots < j_s \leq m$, such that the equalities

$$c_1\mathbf{a}_{j_1} + \dots + c_s\mathbf{a}_{j_s} = 0$$

and

$$c_1 + \dots + c_s = 0,$$

with $c_i \in \mathbb{Q}$, for every $1 \leq i \leq s$, imply that $c_1 = \dots = c_s = 0$. Now we consider the set $\{u_1, u_{j_1}, \dots, u_{j_s}\}$ and assume that there exist $d_0, d_1, \dots, d_s \in \mathbb{Q}$, such that $d_0 + d_1 + \dots + d_s = 0$ and

$$d_0\mathbf{a}_1 + d_1\mathbf{a}_{j_1} + \dots + d_s\mathbf{a}_{j_s} = 0.$$

Looking at the first component of the vector $d_0\mathbf{a}_1 + d_1\mathbf{a}_{j_1} + \dots + d_s\mathbf{a}_{j_s}$, it follows that $d_0 = 0$ and hence, $d_1 + \dots + d_s = 0$ and

$$d_1\mathbf{a}_{j_1} + \dots + d_s\mathbf{a}_{j_s} = 0.$$

By the choice of integers j_1, \dots, j_s , we conclude that $d_1 = \dots = d_s = 0$. Therefore $\text{arank}(I) \geq s + 1 = \text{arank}(I') + 1$. \square

Remark 2.5. It is completely clear from the proof of the Lemma 2.4, that one can consider any arbitrary variable instead of x_1 .

We are now ready to state and prove the main result of this section.

Theorem 2.6. *Let I be a weakly polymatroidal ideal of $S = \mathbb{K}[x_1, \dots, x_n]$. Then we have the following assertions:*

- (i) $\text{sdepth}(I) \geq n - \text{arank}(I) + 1$ and $\text{sdepth}(S/I) \geq n - \text{arank}(I)$.
- (ii) $\text{depth}(S/I) \geq n - \text{arank}(I)$.

Proof. We prove (i) and (ii) simultaneously by induction on n and

$$\sum_{u \in G(I)} \deg(u),$$

where $G(I)$ is the set of minimal monomial generators of I . If $n = 1$ or

$$\sum_{u \in G(I)} \deg(u) = 1,$$

then I is a principal ideal and so we have $\text{arank}(I) = 1$, $\text{sdepth}(I) = n$, $\text{depth}(S/I) = n - 1$ and by [6, Theorem 1.1], $\text{sdepth}(S/I) = n - 1$. Therefore, in these cases, the inequalities in (i) and (ii) are trivial.

We now assume that $n \geq 2$ and

$$\sum_{u \in G(I)} \deg(u) \geq 2.$$

Let $S' = \mathbb{K}[x_2, \dots, x_n]$ be the polynomial ring obtained from S by deleting the variable x_1 and consider the ideals $I' = I \cap S'$ and $I'' = (I : x_1)$. If

$$x_1 \notin \bigcup_{u \in G(I)} \text{Supp}(u_i),$$

then the induction hypothesis on n implies that

$$\text{depth}(S/I) = \text{depth}(S'/I') + 1 \geq (n - 1) - \text{arank}(I') + 1 = n - \text{arank}(I).$$

On the other hand, by [6, Theorem 1.1] and [3, Lemma 3.6], we conclude that $\text{sdepth}(S/I) = \text{sdepth}(S'/I') + 1$ and $\text{sdepth}(I) = \text{sdepth}(I') + 1$. Therefore, using the induction hypothesis on n we conclude that $\text{sdepth}(I) \geq n - \text{arank}(I) + 1$ and $\text{sdepth}(S/I) \geq n - \text{arank}(I)$. Therefore, we may assume that

$$x_1 \in \bigcup_{u \in G(I)} \text{Supp}(u_i),$$

Now $I = I'S' \oplus x_1 I''S$ and $S/I = (S'/I'S') \oplus x_1(S/I''S)$ and therefore by definition of the Stanley depth we have

$$(1) \quad \text{sdepth}(I) \geq \min\{\text{sdepth}_{S'}(I'S'), \text{sdepth}_S(I'')\},$$

and

$$(2) \quad \text{sdepth}(S/I) \geq \min\{\text{sdepth}_{S'}(S'/I'S'), \text{sdepth}_S(S/I'')\}.$$

On the other hand, by applying the depth lemma on the exact sequence

$$0 \longrightarrow S/(I : x_1) \longrightarrow S/I \longrightarrow S/(I, x_1) \longrightarrow 0$$

we conclude that

$$(3) \quad \text{depth}(S/I) \geq \min\{\text{depth}_{S'}(S'/I'S'), \text{depth}_S(S/I'')\}.$$

Using Lemma 2.2 it follows that I'' is a weakly polymatroidal ideal and by Lemma 2.3 we conclude that $\text{arank}(I'') \leq \text{arank}(I)$. Hence our induction hypothesis on

$$\sum_{u \in G(I)} \deg(u)$$

implies that

$$\text{depth}_S(S/I'') \geq n - \text{arank}(I'') \geq n - \text{arank}(I),$$

$$\text{sdepth}_S(S/I'') \geq n - \text{arank}(I'') \geq n - \text{arank}(I),$$

and

$$\text{sdepth}_S(I'') \geq n - \text{arank}(I'') + 1 \geq n - \text{arank}(I) + 1.$$

On the other hand $I'S'$ is a weakly polymatroidal ideal and since

$$x_1 \in \bigcup_{i=1}^s \text{Supp}(u_i),$$

using Lemma 2.4 we conclude that $\text{arank}(I'S') \leq \text{arank}(I) - 1$ and therefore by our induction hypothesis on n we conclude that

$$\begin{aligned} \text{sdepth}_{S'}(I'S') &\geq (n-1) - \text{arank}(I'S') + 1 \geq (n-1) - (\text{arank}(I) - 1) + 1 \\ &= n - \text{arank}(I) + 1, \end{aligned}$$

and similarly $\text{sdepth}_{S'}(S'/I'S') \geq n - \text{arank}(I)$ and $\text{depth}_{S'}(S'/I'S') \geq n - \text{arank}(I)$. Now the assertions follow by inequalities (1), (2) and (3). \square

As an immediate consequence of Proposition 1.2 and Theorem 2.6, we conclude the following result which appeared in [5].

Corollary 2.7. *Let I be a weakly polymatroidal ideal of $S = \mathbb{K}[x_1, \dots, x_n]$ which is generated in a single degree. Then we have the following assertions:*

- (i) $\text{sdepth}(I) \geq n - \ell(I) + 1$ and $\text{sdepth}(S/I) \geq n - \ell(I)$.
- (ii) $\text{depth}(S/I) \geq n - \ell(I)$.

Using Theorem 2.6 we provide an upper bound for the height of associated primes of a weakly polymatroidal ideal.

Corollary 2.8. *Let I be a weakly polymatroidal ideal of $S = \mathbb{K}[x_1, \dots, x_n]$. Then*

$$\max\{\text{ht}(\mathfrak{p}) \mid \mathfrak{p} \in \text{Ass}(S/I)\} \leq \text{arank}(I).$$

Proof. Let $\mathfrak{p} \in \text{Ass}(S/I)$ be given. By [1, Proposition 1.2.13] we have $\text{depth}(S/I) \leq n - \text{ht}(\mathfrak{p})$, while by Theorem 2.6 we have $\text{depth}(S/I) \geq n - \text{arank}(I)$. This implies that $\text{ht}(\mathfrak{p}) \leq \text{arank}(I)$ for every $\mathfrak{p} \in \text{Ass}(S/I)$ and completes the proof of the corollary. \square

3. STANLEY DEPTH OF SQUAREFREE MONOMIAL IDEALS

Let $I \subset S$ be an arbitrary ideal. An element $f \in S$ is *integral* over I , if there exists an equation

$$f^k + c_1 f^{k-1} + \dots + c_{k-1} f + c_k = 0 \quad \text{with } c_i \in I^i.$$

The set of elements \bar{I} in S which are integral over I is the *integral closure* of I . It is known that the integral closure of a monomial ideal $I \subset S$ is a monomial ideal generated by all monomials $u \in S$ for which there exists an integer k such that $u^k \in I^k$ (see [2, Theorem 1.4.2]).

In [8], the author proposed the following conjecture regarding the Stanley depth of integrally closed monomial ideals.

Conjecture 3.1. *Let $I \subset S$ be an integrally closed monomial ideal. Then $\text{sdepth}(S/I) \geq n - \ell(I)$ and $\text{sdepth}(I) \geq n - \ell(I) + 1$.*

In this section we prove that Conjecture 3.1 is true for every squarefree monomial ideal which is generated in a single degree. Indeed we show that for every squarefree monomial ideal I of the polynomial ring S , the inequalities $\text{sdepth}(I) \geq n - \text{rank}(I) + 1$ and $\text{sdepth}(S/I) \geq n - \text{rank}(I)$ hold (see Theorem 3.3).

First we need the following lemma.

Lemma 3.2. *Let I be a squarefree monomial ideal. Then for every $1 \leq j \leq n$ we have $\text{rank}((I : x_j)) \leq \text{rank}(I)$.*

Proof. Let $G(I) = \{u_1, \dots, u_m\}$ be the set of minimal monomial generators of I . Without loss of generality, we may assume that u_1, \dots, u_t are divisible by x_j and u_{t+1}, \dots, u_m are not divisible by x_j , where $0 \leq t \leq m$. Put $v_i = u_i/x_j$, if $1 \leq i \leq t$ and $v_i = u_i$, if $t+1 \leq i \leq m$. For simplicity we assume that \mathbf{a}_i is the exponent vector of u_i and \mathbf{b}_i is the exponent vector of v_i ($1 \leq i \leq m$). To prove the assertion one just note that for every $k \neq j$ and every $1 \leq i \leq m$, the k th component of \mathbf{a}_i and \mathbf{b}_i are the same and for $k = j$, the k th component of \mathbf{b}_i is always zero. \square

We are now ready to state and prove the main result of this section.

Theorem 3.3. *Let I be a squarefree monomial ideal of $S = \mathbb{K}[x_1, \dots, x_n]$. Then $\text{sdepth}(I) \geq n - \text{rank}(I) + 1$ and $\text{sdepth}(S/I) \geq n - \text{rank}(I)$.*

Proof. Let $G(I)$ be the set of minimal monomial generators of I . We prove the assertions by induction on n . If $n = 1$ then I is a principal ideal and so we have $\text{rank}(I) = 1$, $\text{sdepth}(I) = n$ and by [6, Theorem 1.1], $\text{sdepth}(S/I) = n - 1$. Therefore, in this case, there is nothing to prove.

We now assume that $n \geq 2$. Let $S' = \mathbb{K}[x_2, \dots, x_n]$ be the polynomial ring obtained from S by deleting the variable x_1 and consider the ideals $I' = I \cap S'$ and $I'' = (I : x_1)$. If

$$x_1 \notin \bigcup_{u \in G(I)} \text{Supp}(u_i),$$

then by [6, Theorem 1.1] and [3, Lemma 3.6], we conclude that $\text{sdepth}(S/I) = \text{sdepth}(S'/I') + 1$ and $\text{sdepth}(I) = \text{sdepth}(I') + 1$. Therefore, using our induction hypothesis, we conclude that $\text{sdepth}(I) \geq n - \text{rank}(I) + 1$ and $\text{sdepth}(S/I) \geq n - \text{rank}(I)$. Hence we may assume that

$$x_1 \in \bigcup_{u \in G(I)} \text{Supp}(u_i),$$

Now $I = I'S' \oplus x_1 I''S$ and $S/I = (S'/I'S') \oplus x_1(S/I''S)$ and therefore by the definition of Stanley depth we have

$$(1) \quad \text{sdepth}(I) \geq \min\{\text{sdepth}_{S'}(I'S'), \text{sdepth}_S(I'')\},$$

and

$$(2) \quad \text{sdepth}(S/I) \geq \min\{\text{sdepth}_{S'}(S'/I'S'), \text{sdepth}_S(S/I'')\}.$$

Note that the generators of I'' belong to S' . Therefore our induction hypothesis implies that

$$\text{sdepth}_{S'}(S'/I'') \geq (n-1) - \text{rank}(I'')$$

and

$$\text{sdepth}_{S'}(S'/I'') \geq (n-1) - \text{rank}(I'') + 1$$

Using Lemma 3.2 together with [6, Theorem 1.1] and [3, Lemma 3.6], we conclude that

$$\text{sdepth}(S/I'') = \text{sdepth}_{S'}(S'/I'') + 1 \geq (n-1) - \text{rank}(I'') + 1 \geq n - \text{rank}(I),$$

and

$$\text{sdepth}_S(I'') = \text{sdepth}_{S'}(I'') + 1 \geq (n-1) - \text{rank}(I'') + 1 + 1 \geq n - \text{rank}(I) + 1.$$

On the other hand, since

$$x_1 \in \bigcup_{i=1}^s \text{Supp}(u_i),$$

it follows that $\text{rank}(I'S') \leq \text{rank}(I) - 1$ and therefore by our induction hypothesis we conclude that

$$\begin{aligned} \text{sdepth}_{S'}(I'S') &\geq (n-1) - \text{rank}(I'S') + 1 \geq (n-1) - (\text{rank}(I) - 1) + 1 \\ &= n - \text{rank}(I) + 1, \end{aligned}$$

and similarly $\text{sdepth}_{S'}(S'/I'S') \geq n - \text{rank}(I)$. Now the assertions follow by inequalities (1) and (2). \square

As an immediate consequence of Proposition 1.2 and Theorem 3.3 we conclude that Conjecture 3.1 is true for every squarefree monomial ideal which is generated in a single degree.

Corollary 3.4. *Let I be a squarefree monomial ideal of $S = \mathbb{K}[x_1, \dots, x_n]$ which is generated in a single degree. Then $\text{sdepth}(I) \geq n - \ell(I) + 1$ and $\text{sdepth}(S/I) \geq n - \ell(I)$.*

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